JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **161**, 148–175 (1991)

Conditions on the Boundary of the Zero Set and Application to Stabilization of Systems with Uncertainty

GILA FRUCHTER*

Department of Mathematics, Technion-Israel Institute of Technology, Haifa, 32000 Israel

URI SREBRO†

Department of Mathematics, University of California, San Diego, California 92037, and Department of Mathematics, Technion-Israel Institute of Technology, Haifa, 32000 Israel

AND

EZRA ZEHEB†

Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, 32000 Israel

Submitted by C. Foias

Received April 17, 1990

We describe an analytic method for finding the location of the zero set of a vector-valued function which depends on m real variables and n complex parameters. We apply the method to robust stabilization of multivariable linear feedback systems. We find exact measures of the extent of permissible perturbations in the plant and/or the compensator that maintain feedback stability. © 1991 Academic Press, Inc.

I. INTRODUCTION

Various stability and design problems in uncertain control theory can be *reduced* to the problem of *locating the zero set* of certain functions. The concept of zero sets has been recently developed into a tool [1–3] used in solving problems which arise in engineering system theory. It has been successfully used in a variety of areas [1–8], in particular concerning

* Present address: The Dept. of Aerospace, University of California Los Angeles, Los Angeles, CA 90024.

† The second and third authors' work was supported in part by the Technion Fund for the Promotion of Research. The second author acknowledges the H. Kieval Research Fund.

0022-247X/91 $3.00

stability where parameters with uncertainty are involved, and/or where design parameters are searched.

Our purpose is to extend and enhance the zero set method so that it can be applied effectively and more efficiently to vector-valued functions of any dimension; and to cases where the equations obtained by the previous zero set method are not of maximal rank. We then want to solve several robust control problems which cannot be solved by the previous methods.

The following is a very general definition of the zero set, which generalizes earlier definitions, an example of a general robust design problem for a linear time-invariant system, and a reduction of the design problem to the problem of locating the zero set of a certain function.

A Zero Set. Given an open set G in $\mathbb{R}^m$, $m \geqslant 1$, closed sets $K_1, ..., K_n$ in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $K = K_1 \times \cdots \times K_n$ and a continuous function $f: K \times G \to \mathbb{R}^d$, the *zero set* of f relative to K and G is defined by

$$V = \{s \in G: f(A, s) = 0 \text{ for some } A \text{ in } K\}.$$

If we let $f_A(s) = f(A, s)$, $A \in K$, $s \in K$, then

$$V = \bigcup_{A \in K} f_A^{-1}(0).$$

A Design Problem. Consider the standard linear time-invariant continuous multivariable system

$$\dot{x} = Ax + Bu$$

$$y = Cx,$$

where A, B, and C are given constant matrices, and u, y, and x are the input, the output, and the state vectors, respectively. Suppose now that some or all entries of A, B, and C, say r_i, vary in intervals $K_i = [\alpha_i, \beta_i]$, $i = 1, ..., n-1$, respectively. So $A = A(r)$, $B = B(r)$, and $C = C(r)$ where $r = (r_1, ..., r_{n-1}) \in Q = \prod_{i=1}^{n-1} K_i$.

Our problem, which is one of the central design problems in robust control, is to find *all* constant gain output feedback matrices H (possibly with prescribed designer constraints on the gains) which ensure relative robust stability of the closed-loop system. For this purpose the characteristic values of the closed-loop system should be confined robustly to a given desired open set D in the left half complex plane.

Reduction to a Problem of Zero Set Location. The state equations for the family of closed-loop systems, mentioned above, corresponding to all r in Q, have the form

$$\dot{x} = Ax + B(u + Hy) = (A + BHC)x + Bu,$$

$A = A(r)$, $B = B(r)$, and $C = C(r)$. Let m be the number of entries of H. Then H can be represented by a vector $s = (s_1, ..., s_m)$. In order to get a complete solution to the above design problem one has to determine the set S of all points $s = (s_1, ..., s_m)$ in a given set G (of the designer constraints on the gains) in $\mathbb{R}^m$ for which all roots p of the characteristic polynomial

$$f(p, r, s_1, ..., s_m) = \det(pI - A - BHC)$$

lie in D for any value of r in Q. Then the desired set S is given by

$$S = \{s \in G \subset \mathbb{R}^m : f(p, r, s_1, ..., s_m) \neq 0,\ \forall p \in \mathbb{C} \backslash D \ \&\ \forall r \in Q\}.$$

Denote $K_n = \mathbb{C} \backslash D$ and $K = Q \times K_n = \prod_{i=1}^{n} K_i$. Then the complement of S in G becomes

$$V = G \backslash S = \{s \in G : \exists (p, r) \in K \text{ such that } f(p, r, s_1, ..., s_m) = 0\},$$

which is recognized as the zero set of f relative to K and G. In conclusion, the problem of finding all constant gain output feedback matrices H (possibly with prescribed designer constraints on the gains) which robustly stabilize the system is reduced to the problem of finding the zero set V of f relative to K and G. Note that there is a similar reduction for the discrete case.

Locating the Zero Set. Since many control problems, like the one mentioned above, can be solved completely once a certain zero set is located, it is very important to find good methods for locating zero sets. The starting point in the existing methods is to locate the boundary ∂V of the zero set V. This is done in [1, 3] and here by adding to the equations $f(A, s) = 0$ new equations which reflect necessary conditions which points s of ∂V must satisfy. In [1] it is done for complex-valued holomorphic functions $f(A, s)$, $A \in K$, $s \in \mathbb{C}$, in [3] for complex-valued functions $f(A, s)$, $A \in K$, $s \in G \subset \mathbb{R}^m$, which satisfy some very mild smoothness conditions, and now it is generalized to functions f from $K \times G$ into $\mathbb{R}^d$, where K is as before, $G \subset \mathbb{R}^m$, and d is any integer $\geqslant 1$. Note that earlier only $d = 2$ was considered. The results of [3] and their immediate generalization, see the main theorem in [3] and in Section III below, yield in the generic case, i.e., when the equations of the main theorem are independent, an $(m-1)$-dimensional set in G which contains ∂V. In such cases the method of [3], for locating ∂V and then the zero set V, is effective. It may happen, however, that the equations obtained by the main theorem of [3] are not independent, as illustrated here by a special case of the design problem formulated above. In order to handle such cases and in order to make the zero set method more efficient, convenient, and applicable in a wider class of problems, we provide here (see Theorem 1 and Theorem 2 of Section III) new methods of generating new equations for ∂V. The new equations obtained by Theorems 1 and 2 of Section III can be used with part or with all the equations given by the main theorem here or in [3], to resolve the case of dependent equations and/or to reduce the computational complexity in locating ∂V by taking advantage of the flexibility and large choice of pertinent equations. Software packages such as the symbolic mathematical software "mumath" are very useful in this respect. The computations, concerned with the application of our analytical method, were made by a personal computer.

Solution of the Design Problem by Zero Set Location. We will consider here a special case of the general design problem formulated above, which cannot be solved by the methods of [3] without additional equations for ∂V. We will apply the main theorem (Section II) and the new Theorems 1 and 2 to find a complete solution by locating a certain zero set. This example, though simple, indicates the solution of the general design problem and illustrates the computational advantage of the new equations which Theorems 1 and 2 provide.

This paper consists of five sections and an Appendix. In Section II we introduce the main theorem and the procedure of determining the zero set extended for scalar and vector-valued functions. In Section III we derive and present the new method, Theorems 1 and 2, of generating new equations for the boundary of zero sets, and provide examples. In Section IV we consider and solve a special case of the design problem described above, as well as a design centering problem. We conclude with Section V. Some of the computations are detailed in the Appendix. A detailed discussion of the various possibilities of combining the equations of the main theorem with those of Theorem 1 and 2, in locating the zero set of a function, will be provided in a separate note of this issue.

Considerable effort in recent years has been devoted to this kind of problem by several authors, among them [2–4, 9–36]. Applying the mathematical results of this paper, we are able to solve very general and difficult control problems, like this formulated above, which could not be solved previously. The results of this paper will enlarge the applicability of the tool introduced in [1–3] to still further problems in system theory and especially in control theory. These results were already applied in [37, 38] to design problems in absolute robust stabilization of nonlinear multivariable control systems under uncertainty conditions.

II. The Main Theorem for Scalar and Vector-Valued Functions and the Procedure of Determining the Zero Set

A. *The Zero Set*

Let $K = K_1 \times \cdots \times K_n$ be a set in $\bar{\mathbb{C}}^n = \bar{\mathbb{C}} \times \cdots \times \bar{\mathbb{C}}$ (n times), where each K_i is a closed set in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let G be an open set in $\mathbb{R}^m$ and let $f\colon K \times G \to \mathbb{R}^d \cup \{\infty\}$ be a continuously differentiable function. The last assumption means that $K \times G$ has an open neighborhood, where $f(A, s) = (f_1(A, s), ..., f_d(A, s))$ has continuous partial derivatives with respect to all coordinates x_i, y_i, and s_i where $A_i = x_i + jy_i$, $A = (A_1, ..., A_n)$ is a point in K, and $s = (s_1, ..., s_m)$ is a point in G. The zero set of $f = (f_1, ..., f_d)$ relative to K and G is then defined by

$$V = \{s \in G \subset \mathbb{R}^m \colon \exists A \in K \text{ such that } f(A, s) = 0\}. \tag{1}$$

In other words, $s \in V$ if and only if $f(A, s) = 0$ for some point A in K. In [3] we considered the case when $d = 2$. Note that a scalar complex-valued function is essentially equivalent to a vector-valued *real* function with $d = 2$.

Our main purpose in this section is to adopt the algorithm derived in [3] to the problem of locating the zero set V of a vector-valued function. This objective will be carried out by extending the main theorem to the general case.

LEMMA. *V is a closed set relative to G.*

Proof. See [3, Lemma 1].

B. *Necessary Conditions on the Relative Boundary of the Zero Set*

THE MAIN THEOREM. *Let $K = K_1 \times \cdots \times K_n$ be a set in $\bar{\mathbb{C}}^n = \bar{\mathbb{C}} \times \cdots \times \bar{\mathbb{C}}$ (n times), where each K_i is a closed set in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ whose boundary ∂K_i is a finite union of piecewise-smooth simple curves and piecewise-smooth closed simple curves. Let G be an open set in $\mathbb{R}^m$. Let $f\colon K \times G \to \mathbb{R}^d \cup \{\infty\}$ be a continuously differentiable function and let V be the zero set of f relative to K and G as defined in* (1).

Suppose that s^0 is a point in the boundary of V relative to G and $A^0 = (A_1^0, ..., A_n^0)$ is a point in K such that $f(A^0, s^0) = 0$. Suppose that for $i = 1, ..., l$, $0 \leqslant l \leqslant n$, A_i^0 belongs to the boundary of a connected component of K_i having a parametric representation

$$A_i = A_i(\theta_i), \qquad 0 \leqslant \theta_i \leqslant 1$$

and that for $i = l + 1, ..., n$, $A_i^0 \in \operatorname{int} K_i$.

Suppose that $\theta_1^0, \ldots, \theta_l^0$ *are numbers in the open intervals* $(0, 1)$ *such that*

$$A_i(\theta_i^0) = A_i^0$$

and x_i^0, y_i^0 *are real numbers such that for* $i = l+1, \ldots, n$, $A_i^0 = x_i^0 + jy_i^0$.

Let $\delta \in (0, 1)$ *be such that* $(\theta_i^0 - \delta, \theta_i^0 + \delta) \subset (0, 1)$ *for* $i = 1, \ldots, l$ *and* $(x_i^0 - \delta, x_i^0 + \delta) \times (y_i^0 - \delta, y_i^0 + \delta) \subset K_i$ *for* $i = l+1, \ldots, n$.

Denote

$$\xi^0 = (\xi_1^0, \ldots, \xi_r^0) = (\theta_1^0, \ldots, \theta_l^0, x_{l+1}^0, y_{l+1}^0, \ldots, x_n^0, y_n^0)$$

and

$$\xi = (\xi_1, \ldots, \xi_r) = (\theta_1, \ldots, \theta_l, x_{l+1}, y_{l+1}, \ldots, x_n, y_n)$$

for each point $\xi \in \mathbb{R}^r$ *such that*

$$|\xi_i - \xi_i^0| < \delta, \qquad i = 1, \ldots, r.$$

Then, at points (A^0, s^0), *where the derivatives of* $A_i(\theta_i)$ *exist, we have the relations*

$$f_i(A^0, s^0) = 0, \qquad i = 1, \ldots, d \tag{2}$$

$$\frac{\partial(f_1, \ldots, f_d)}{\partial(\xi_{i_1}, \ldots, \xi_{i_d})}(A^0, s^0) = 0, \qquad 1 \leqslant i_1 < \cdots < i_d \leqslant r. \tag{3}$$

At points where any of the coordinates is ∞, *the differentiability of* f *and the conditions* (2) *and* (3) *should be evaluated after suitable changes of variables of the form* $z \to 1/z$ *are performed.*

Remark. 1. If for a certain choice of l, $d > r$, then (3) is vacuous. In this case, however, (3) is not needed for the procedure of determining the zero set.

Remark 2. In [3, The Main Theorem] we stated and proved this theorem for $d = 2$, i.e., for the case when f is complex-valued.

The proof of the main theorem is as in [3, II-D], where the principal tool is the implicit function theorem for functions with values in $\mathbb{R}^d$.

C. *Analysis of the Main Theorem*

Let K, G, $f = (f_1, \ldots, f_d)$, V, (A^0, s^0), $\xi^0 = (\xi_1^0, \ldots, \xi_r^0)$, and $\xi = (\xi_1, \ldots, \xi_r)$ with $A = A(\xi)$ and $A^0 = A(\xi^0)$ as in the main theorem.

The relations (2) and (3) are necessary conditions for points s^0, where differentiability exists, to belong to the relative boundary $\partial_G V = \partial V \cap G$ of the zero set V.

We assumed in the main theorem that the first l coordinates A_i of A belong to ∂K_i, $i=1, ..., l$, and the other coordinates belong to int K_i, $\hat{i}=l+1, ..., n$, but there is no loss of generality in this assumption, since it is always possible to rearrange the order of the coordinates. Therefore, the main theorem may be applied to any point A^0 in K where l of the coordinates are on the boundaries ∂K_i and any point s^0 in $\partial_G V$ where differentiability exists and for which $f(A^0, s^0)=0$. We now consider every possible l. For a given l, the point A depends on the $r=l+2(n-l)=2n-l$ real parameters

$$(\xi_1, ..., \xi_r)=(\theta_1, ..., \theta_l, x_{l+1}, y_{l+1}, ..., x_n, y_n).$$

Therefore, the equations corresponding to (2) and (3) depend on $m+r$ real parameters $\xi_1, ..., \xi_r, s_1, ..., s_m$. We now compute for each r the maximal number of independent equations obtained from the relations (2) and (3).

CLAIM. *Let $\xi_1, ..., \xi_r, f_1, ..., f_d$ and (A^0, s^0) be as in the main theorem. The maximal number of independent equations in the collection* (3) *is* $r-d+1$.

Proof. The proof is trivial for $d=1$. If $d>1$, let us consider the matrix

$$J=(J_{ij})_{j=1,...,r}^{i=1,...,d},$$

where

$$J_{ij}=\frac{\partial f_i}{\partial \xi_j}(A^0, s^0).$$

If every $(d-1)$ columns in the matrix J are linearly dependent, then the number of independent equations in (3) is zero, and in this case we have nothing to prove. Therefore, let us suppose that there are $(d-1)$ linearly independent columns in J. With no loss of generality, we may assume that they are the first $(d-1)$ columns. Now, let us consider the following $(r-d+1)$ equations

$$\frac{\partial(f_1, ..., f_d)}{\partial(\xi_1, ..., \xi_{d-1}, \xi_k)}(A^0, s^0)=0, \qquad k=d, d+1, ..., r \tag{4}$$

from the collection (3). The proof of our claim will be implied from the equivalence of the collections (3) and (4). Indeed from (4) we obtain that the first $(d-1)$ columns of J generate the column space of J. But these $(d-1)$ columns are linearly independent, therefore the dimension of the column space of J is $d-1$. This implies (3) and hence the equivalence of (3) and (4).

COROLLARY. *The maximal number of independent equations corresponding to the relations* (2) *and* (3) *in the main theorem is* $r+1$.

In summary, for each r the relations (2) and (3) yield at most $r+1$ independent equations in $r+m$ real unknowns. Let us denote these equations by

$$h_i(A^0, s^0) = 0, \qquad i = 1, ..., r+1 \tag{5}$$

and let $h = (h_1, ..., h_{r+1})$. Suppose that h is a continuously differentiable function in some neighborhood of (A^0, s^0) and in addition

$$\operatorname{rank} h'(A^0, s^0) = r+1, \tag{6}$$

where $h'(A^0, s^0)$ denotes the Jacobian matrix of h at (A^0, s^0). The last assumption means that the number of independent equations in (5) is maximal. In this case, the difference between the number of unknowns and the number of equations is $m-1$. Hence, the set of points in $\mathbb{R}^m$ satisfying these equations is $(m-1)$-dimensional and depends of course on the particular choice of l and on the choice of connected components of the sets $\partial K_1, ..., \partial K_l$, respectively. The same procedure can be repeated for any choice of indices $i_1, ..., i_l$ from the set $\{1, ..., n\}$ instead of $1, ..., l$ and any choice of connected components of the corresponding boundaries. In the next section we will consider the union of these sets.

D. *The Procedure of Determining the Zero Set*

The extension of the main theorem to the general case makes it possible for us to adopt the algorithm of determining the zero set V, presented in [3], to the general case of vector-valued functions. As in the case of complex-valued functions (the case treated in [3]), the procedure of determining the zero set relies on the assumption that the number of independent equations corresponding to the relations (2) and (3) in the main theorem is maximal. In this case, these equations yield an $(m-1)$-dimensional set L. The case where (2) and (3) do not yield an $(m-1)$-dimensional set will be treated in Sections III and IV.

The procedure of determining the zero set is quite similar to the one described in detail in [3, Sect. II] for the case $d=2$. For the sake of completeness we outline the procedure in the following. We conclude with an illustrative example for the case $d=3$. The case $d=1$ is illustrated in [37], several times, by applying the method to solve a variety of engineering problems.

Let f, K, and G be as in the main theorem. For each l in $\{0, ..., n\}$, choose a subset of l different indices from the set $\{1, 2, ..., n\}$. For each index i in this subset, pick a connected component of ∂K_i. Finally write down the system of $2n-l+1$ (or $r+1$) equations corresponding to (2) and (3). The number of unknowns in this system of $2n-l+1$ equations is $2n-l+m$. If (2) and (3) yield a maximal number of independent equations, each of the unknowns can be represented as a function of $m-1$ real parameters $t_1, ..., t_{m-1}$, $0<t_i<1$, $i=1, ..., m-1$, so that (2)–(3) are satisfied identically in $t_1, ..., t_{m-1}$. In particular,

$$\{(s_1(t_1, ..., t_{m-1}), ..., s_m(t_1, ..., t_{m-1})): 0<t_1<1, i=1, ..., m-1\}$$

is an $(m-1)$-dimensional set in $\mathbb{R}^m$. We do the same for all possible choices of l, $0 \leqslant l \leqslant n$, of the indices $i_i, ..., i_l$ from the set $\{1, ..., n\}$ and of connected components of $\partial K_{i_1}, ..., \partial K_{i_l}$, respectively. We assume that for each of the choices, (2)–(3) yield an $(m-1)$-dimensional set in $\mathbb{R}^m$ and we denote by $\mathscr{L}_0$ the union of all these $(m-1)$-dimensional sets.

Next, consider a finite set of points, say $b_1, ..., b_p$, in $\bigcup_{i=1}^n \partial K_i$ which correspond to the points where the derivatives of $A_i(\theta_i)$ do not exist. In the sequel, we will label such points "bad" points. Suppose that $b_1 \in \partial K_1$. By substituting $A_1 = b_1, f(A, s) = 0$ reduces to a (vector) equation in the (complex) unknowns $A_2, ..., A_n$ and s, $s \in \mathbb{R}^m$. We apply the main theorem to the new equation at the point $(b_1, A_2, ..., A_n)$ and obtain $(m-1)$-dimensional sets in $\mathbb{R}^m$ in a way similar to the previous procedure for $\mathscr{L}_0$. Next we repeat the procedure for b_2, b_3, etc., up to b_p. Denote by $\mathscr{L}_1$ the union of all these $(m-1)$-dimensional sets for $b_1, ..., b_p$.

Next, we substitute in $f(A, s) = 0$ two bad points, say b_{n_i} for A_i and b_{n_k} for A_k, such that $i \neq k$, and apply the main theorem. We assume that again we obtain $(m-1)$-dimensional sets for each choice of i and k, and denote by $\mathscr{L}_2$ the union of all these sets. Next, we choose three bad points, then four bad points, etc., and obtain $\mathscr{L}_3, ..., \mathscr{L}_p$. Let

$$\mathscr{L} = \bigcup_{i=0}^{p} \mathscr{L}_i.$$

Then, since each $\mathscr{L}_i$ is $(m-1)$-dimensional, so is $\mathscr{L}$ and

$$\partial_G V \subset \mathscr{L}.$$

Suppose that for a certain choice of indices and boundary components, (2)–(3) yield an $(m-1)$-dimensional set C. Then $C \subset \mathscr{L}$ but not every point of C necessarily belongs to V. We can determine the points of C which belong to V as follows.

The system (2)–(3) yields a parametric representation $A_i = A_i(t_1, ..., t_{m-1})$ of each of the parameters $A_i = x_i + jy_i$ or $A_i(\theta_i(t_1, ..., t_{m-1}))$, $0<t_j<1$, $j=1, ..., m-1$, and a parametric representation $s_1(t_1, ..., t_{m-1}), ..., s_m(t_1, ..., t_{m-1})$ for every point $s=(s_1, ..., s_m)$ in G. Now a point $s(t_1, ..., t_{m-1})$ in C belongs to V if $A_i(t_1, ..., t_{m-1}) \in K_i$ for all i for which

x_i, y_i, and θ_i are involved in (2)–(3). By doing the same for all such sets C in $\mathscr{L}$, we can decide for each i, $i = 0, ..., p$, which part of $\mathscr{L}_1$ is included in V. Let $L_0, L_1, ..., L_p$ denote the subsets of $\mathscr{L}_0$, $\mathscr{L}_1, ..., \mathscr{L}_p$, respectively, which are included in V and let

$$L = \bigcup_{i=1}^{p} L_i.$$

Then

$$\partial_G V \subset L \subset V.$$

Suppose now that $G \backslash L$ has only finitely many connected components $D_1, ..., D_q$. Note that each component D_i is an open connected set in G and that $D_i \cap \partial V = \varnothing$. Therefore, either $D_i \subset V$ or else $D_i \cap V = \varnothing$. In order to decide which of the two cases holds, we pick an arbitrary point s^i in D_i and consider the equation

$$f(A_1, ..., A_n, s^i) = 0$$

in the unknowns $A_1, ..., A_n$. Let G_1 be an open set in $\mathscr{C}$ such that $G_1 \supset K_1 \backslash \{\infty\} = K_1 \cap \mathbb{C}$ and in which f is continuously differentiable. Consider the zero set

$$V_1 = \left\{ A_1 \in G_1 : \exists (A_2, ..., A_n) \in \prod_{i=2}^{n} K_i \text{ s.t. } f(A_1, ..., A_n, s^i) = 0 \right\}.$$

Note that we have again the original problem, where now the number of complex parameters has decreased by 1 and V_1 is a two-dimensional set. We follow the above procedure and find a new set L^1 such that

$$\partial_{G_1} V_1 \subset L^1 \subset V_1.$$

If $L^1 \cap K_1 \neq \varnothing$, then $s^i \in V$; and $D_i \subset V$. Otherwise we pick an arbitrary point A_i^i in each open connected component D_1^i of $G_1 \backslash L^1$, and consider the equation

$$f(A_1^i, ..., A_n, s^i) = 0$$

in the $n - 1$ unknowns $A_2, ..., A_n$; therefore, we again have the original problem in a lower dimension. Continuing in this way, we complete the procedure of finding the components D_i of $G \backslash L$ which are included in V.

Let $D_1, ..., D_k$ be the connected components of $G \backslash L$ which are included in V; then

$$V = \bigcup_{i=1}^{k} D_i \cup L.$$

III. New Equations for the Boundary of the Zero Set

Let V be the zero set of f relative to K and G as described in the main theorem, see II-B above. The main theorem provides equations for the boundary of V in G which together with the d equations $f=0$ give in the generic case an $(m-1)$-dimensional set L in G which contains the boundary of V. If, however, the equations are dependent, L is not of dimension $m-1$ and the procedure of locating V cannot be carried out along the lines described in II-D. The following two theorems provide new equations for the boundary of the zero set which eventually yield a set L of the right dimension. The new equations can be used even in the case where the original equations are independent. This flexibility in selecting the equations may reduce the computational complexity.

Let $f(A, s)$, where $A=A(\xi)$ and $\xi=(\xi_1, ..., \xi_r)$, be as in II-B. In the following theorem we assume that the d equations $f=0$ are independent, but $f=0$ together with the equations obtained from the main theorem are not independent. We also assume that p of the variables and parameters can be solved in terms of the other $m+r-p$ variables and parameters. Then $p \geqslant d$. Furthermore, we assume that q out of p are parameters ξ_i and $p-q$ are variables s_i. The new equations in the theorem are given in terms of jacobians of the $p-q$ s_i's with respect to the independent ξ_i's. Note that the main theorem implies that $p-q \geqslant 1$.

Theorem 1. *Let K, G, $f=(f_1, ..., f_d)$, V, (A^0, s^0), $\xi^0=(\xi_1^0, ..., \xi_r^0)$, and $\xi=(\xi_1, ..., \xi_r)$ with $A=A(\xi)$ and $A^0=A(\xi^0)$ as in the main theorem.*

Let p and q be integers such that $d \leqslant p \leqslant r$ and $0 \leqslant q \leqslant p-1$. Suppose that there is a $(m+r-p)$-dimensional neighborhood $N=N_1 \times N_2$ of the point $(\xi^{1,0}, s^{1,0})=(\xi_{q+1}^0, ..., \xi_r^0, s_{p\ \ q+1}^0, ..., s_m^0)$ and a vector-valued function $\varphi: N \to \mathbb{R}^p$, such that

(i) *φ is continuously differentiable on N*

(ii) $\varphi(\xi^{1,0}, s^{1,0})=(\xi_1^0, ..., \xi_q^0, s_1^0, ..., s_{p-q}^0)$

(iii) $f(A(\varphi^1(\xi^1, s^1), \xi^1), \varphi^2(\xi^1, s^1), s^1)=0$ *for every (ξ^1, s^1) in N, where*

$$(\xi^1, s^1)=(\xi_{q+1}, ..., \xi_r, s_{p\ \ q+1}, ..., s_m)$$

and

$$\varphi=(\varphi^1, \varphi^2)=(\varphi_1, ..., \varphi_q, \varphi_{q+1}, ..., \varphi_p).$$

Then at points $(\xi^{1,0}, s^{1,0})$ we have, in addition to (ii), *the equations*

$$\frac{\partial(\varphi_{q+1}, ..., \varphi_p)}{\partial(\xi_{i_1}, ..., \xi_{i_{p-q}})}(\xi^{1,0}, s^{1,0})=0, \qquad q+1 \leqslant i_1 < \cdots < i_{p\ \cdot q} \leqslant r. \tag{7}$$

We need the following lemma for the proof.

LEMMA 1. *With the assumptions and notations of Theorem* 1, *let*

$$G_1 = \{s \in G: s^1 = (s_{p-q+1}, \dots, s_m) \in N_2\}$$

and $g: N_1 \times G_1 \to \mathbb{R}^{p-q}$, $g = (g_1, \dots, g_{p-q})$, *be such that*

$$g(\xi^1, s) = \varphi^2(\xi^1, s^1) - (s_1, \dots, s_{p-q}).$$

Let also

$$V_1 = \{s \in G_1 : \exists \xi^1 \in N_1 \text{ such that } g(\xi^1, s) = 0\}$$

be the zero set of g *relative to* N_1 *and* G_1.

Then $V_1 \subset V$ *and* $s^0 \in \partial_{G_1} V_1$.

Proof. Let $s \in V_1$ then there is $\xi^1 \in N_1$ such that

$$g(\xi^1, s) = \varphi^2(\xi^1, s^1) - (s_1, \dots, s_{p-q}) = 0,$$

where $s^1 \in N_2$ and therefore

$$s = (\varphi^2(\xi^1, s^1), s^1).$$

Since $(\xi^1, s^1) \in N$, $A(\varphi^1(\xi^1, s^1), \xi^1) \in K$, and

$$f(A(\varphi^1(\xi^1, s^1), \xi^1), \varphi^2(\xi^1, s^1), s^1) = 0.$$

Hence $s = (\varphi^2(\xi^1, s^1), s^1)$ belongs to V and therefore $V_1 \subset V$. By (ii) of the theorem and the definitions of g and V_1 it follows that $s^0 \in V_1$. Hence $s^0 \in \partial_{G_1} V_1$. This completes the proof of our lemma.

Proof of Theorem 1. Let G_1, g, and V_1 be as in Lemma 1. By Lemma 1, $s^0 \in \partial_{G_1} V_1$, and by the main theorem applied to g

$$\frac{\partial(g_1, \dots, g_{p-q})}{\partial(\xi_{i_1}, \dots, \xi_{i_{p-q}})}(\xi^{1,0}, s^0) = 0, \qquad q+1 \leqslant i_1 < \cdots < i_{p-q} \leqslant r.$$

But

$$\frac{\partial(g_1, \dots, g_{p-q})}{\partial(\xi_{i_1}, \dots, \xi_{i_{p-q}})}(\xi^{1,0}, s^0) = \frac{\partial(\varphi_{q+1}, \dots, \varphi_p)}{\partial(\xi_{i_1}, \dots, \xi_{i_{p-q}})}(\xi^{1,0}, s^{1,0}).$$

Thus, the theorem follows.

The above theorem is now illustrated by an example which arises in a certain engineering problem.

EXAMPLE 1. Let $K = K_1 \times K_2$ where

$$K_1 = \{A_1 \in \mathbb{R}\colon 1 \leqslant A_1 \leqslant 2\}$$
$$K_2 = \{A_2 \in \mathbb{C}\colon \operatorname{Re} A_2 \geqslant 0\} \cup \{\infty\},$$

and let $f\colon K \times \mathbb{R}^2 \to \bar{\mathbb{C}}$ be defined by

$$f(A_1, A_2, s_1, s_2) = A_2^3 + s_1 A_2^2 + (s_2 - 5s_1 - 13)\, A_2 + s_2 A_1.$$

Let V be the zero set of f and take $A_2 \in \operatorname{int} K_2$ in the main theorem. The objective is to find the set of points (s_1, s_2) on ∂V corresponding to this case. Note that

$$\xi_1 \in (1, 2)$$
$$\xi_2 > 0$$
$$\xi_3 \in \mathbb{R}.$$

Let f_1 and f_2 be the real and imaginary parts, respectively, of f, i.e., $f = (f_1, f_2)$. Then

$$\begin{aligned} &f_1(A(\xi_1, \xi_2, \xi_3), s_1, s_2) \\ &\quad = \xi_2^3 - 3\xi_3^2\xi_2 + s_1\xi_2^2 - s_1\xi_3^2 + (s_2 - 5s_1 - 13)\,\xi_2 + s_2\xi_1 = 0 \\ &f_2(A(\xi_1, \xi_2, \xi_3), s_1, s_2) \\ &\quad = 3\xi_2^2\xi_3 - \xi_3^3 + 2\xi_2\xi_3 s_1 + (s_2 - 5s_1 - 13)\xi_3 = 0. \end{aligned}$$

This problem arose when we used our method of "zero sets location" (based on the main theorem), to robustly stabilize by output feedback a continuous system under uncertainty conditions [37]. The function f represents the characteristic polynomial, s_1, s_2 are the searched design parameters of the feedback, A_1 represents the uncertainty parameter, and A_2 represents the complex frequency.

Applying the main theorem, we obtain that the system (3) is reduced here to

$$\begin{aligned} \frac{\partial(f_1, f_2)}{\partial(\xi_2, \xi_3)} &= (3\xi_2^2 - 3\xi_3^2 + 2\xi_2 s_1 + s_2 - 5s_1 - 13)^2 \\ &\quad + (6\xi_2\xi_3 + 2\xi_3 s_1)^2 = 0. \end{aligned}$$

Solving the last three equations and using the notations of Theorem 1, one can obtain that

$$(\varphi(\xi_2, s_1)) = (\varphi_1(\xi_2, s_1), \varphi_2(\xi_2, s_1), \varphi_3(\xi_2, s_1)) = (\xi_1, \xi_3, s_2)$$
$$= \left(\frac{2\xi_2^3 + s_1\xi_2^2}{5s_1 + 13 - 2s_1\xi_2 - 3\xi_2^2}, 0, 5s_1 + 13 - 2s_1\xi_2 - 3\xi_2^3\right),$$

satisfies these equations in each neighborhood of (ξ_2, s_1) in which $s_2 \neq 0$. Therefore we are in the case where the main theorem *cannot* yield a one-dimenional set of points (s_1, s_2) in $\mathbb{R}^2$. However, applying Theorem 1 we can add another condition

$$\frac{\partial\varphi_3}{\partial\xi_2} = -2s_1 - 6\xi_2 = 0;$$

and obtain that the set of points (s_1, s_2) on ∂V corresponding to this case is included, when $s_2 \neq 0$, in

$$L_0^1 = \{s \in \mathbb{R}^2 \colon s_2 = \tfrac{1}{3}s_1^2 + 5s_1 + 13, \ -6.68 < s_1 < -3.7\}.$$

If $s_2 = 0$ it is easy to see that these points are included in

$$L_0^2 = \{-2(5 \pm \sqrt{12}), 0\}.$$

In conclusion the required set is included in

$$L_0^3 = L_0^1 \cup L_0^2.$$

End of Example 1.

Again, let $f(A, s)$, where $A = A(\xi)$ and $\xi = (\xi_1, ..., \xi_r)$, be as in II–B. In Theorem 1 we assumed that rank $f'(A(\xi), s) = d$. We now relax the assumptions and assume that rank $f'(A(\xi), s) \geqslant 1$. Note that if

$$1 \leqslant \operatorname{rank} f'(A(\xi), s) < d,$$

then

$$\frac{\partial(f_1, ..., f_d)}{\partial(\xi_{i_1}, ..., \xi_{i_d})}(A, s) \equiv 0$$

for every d indices $i_1, ..., i_d$ from the set $\{1, ..., r\}$ and (3) of the main theorem is vacuous and only $f = 0$ is left. In this case Theorem 1, where $d \leqslant p$ is assumed, is not applicable.

The following theorem applies in all cases with no restriction on rank $f'(A(\xi), s)$.

Let k and q be integers such that $1 \leqslant k \leqslant d-1$ and $d-k \leqslant q \leqslant r-1$. Suppose that by solving a system of q equations, where $d-k$ of them are from $f=0$ and $q-(d-k)$ equations are from (3) of the main theorem, we can express q of the parameters ξ_i, which after rearrangement are denoted by $\xi_1, ..., \xi_q$, in terms of all other $m+r-q$ variables and parameters. Note that $q \leqslant r$. If $q=r$ then each ξ_i is expressed as a function of s_i's and by substituting in one of the k remaining equations from $f=0$, we get the desired $(m-1)$-dimensional set. So, we assume $q<r$. The new equations, which the following theorem provides for this situation, are obtained by substituting the expressions for the $\xi_1, ..., \xi_q$ in the k equations of $f=0$ which were not used and by utilizing certain new jacobians.

THEOREM 2. *Let K, G, $f=(f_1, ..., f_d)$, V, (A^0, s^0), $\xi^0=(\xi_1^0, ..., \xi_r^0)$, and $\xi=(\xi_1, ..., \xi_r)$ with $A=A(\xi)$ and $A^0=A(\xi^0)$ as in the main theorem, with $d \geqslant 2$.*

Let k and q be integers such that $1 \leqslant k \leqslant d-1$ and $d-k \leqslant q \leqslant r-1$. Suppose that there is a $(m+r-q)$-dimensional neighborhood $N=N_1 \times N_2$ of the point $(\xi^{1,0}, s^0)=(\xi_{q+1}^0, ..., \xi_r^0, s^0)$ and a vector-valued function $\varphi\colon N \to \mathbb{R}^q$, such that

(i) *φ is continuously differentiable on N*

(ii) *$\varphi(\xi^{1,0}, s^0)=(\xi_1^0, ..., \xi_q^0)$*

(iii) *$v(A(\varphi(\xi^1, s), \xi^1), s)=0$ for every (ξ^1, s) in N, where*

$$\xi^1=(\xi_{q+1}, ..., \xi_r)$$

and

$$v=(f_{k+1}, ..., f_d).$$

Let $g\colon N \to \mathbb{R}^k$ be defined by

$$g(\xi^1, s)=u(A(\varphi(\xi^1, s), \xi^1), s)$$

where $u=(f_1, ..., f_k)$ and $g=(g_1, ..., g_k)$.

Then at points $(\xi^{1,0}, s^0)$ we have

$$g_j(\xi^{1,0}, s^0)=0, \qquad j=1, ..., k \tag{8}$$

$$\frac{\partial(g_1, ..., g_k)}{\partial(\xi_{i_1}, ..., \xi_{i_k})}(\xi^{1,0}, s^0)=0, \qquad q+1 \leqslant i_1 < \cdots < i_k \leqslant r. \tag{9}$$

We need the following lemma for the proof.

LEMMA 2. *With the assumptions and notations of Theorem 2, let*

$$V_1 = \{s \in N_2 \subset G: \exists \xi^1 \in N_1 \text{ such that } g(\xi^1, s) = 0\}$$

be the zero set of g relative to N_1 and N_2.

Then $V_1 \subset V$ and $s^0 \in \partial_{N_2} V_1$.

Proof. Let $s \in V_1$. Then there is $\xi^1 \in N_1$ such that

$$g(\xi^1, s) = u(A(\varphi(\xi^1, s), \xi^1), s) = 0. \tag{10}$$

Since $s \in N_2$ it follows that $(\xi^1, s^1) \in N$, $A(\varphi(\xi^1, s), \xi^1) \in K$, and

$$v(A(\varphi(\xi^1, s), \xi^1), s) = 0. \tag{11}$$

From (10) and (11) we obtain that $s \in V$ and therefore $V_1 \subset V$. From (ii) of the theorem and the fact that $u(A(\xi^0), s^0) = 0$ we obtain that $s^0 \in V_1$ and hence $s^0 \in \partial_{N_2} V_1$. This completes the proof of our lemma.

Proof of Theorem 2. Let V be the zero set of g relative to N_1 and N_2, as in Lemma 2. By Lemma 2 the main theorem is applicable to g at $(\xi^{1,0}, s^0)$ and hence (8) and (9) follow.

Remark 3. If for a certain case $k > r - q$, then (9) is vacuous. In this case, however, it is also not needed.

The above theorem is now illustrated by an example which arises in a certain engineering problem.

EXAMPLE 2. Let $K = K_1 \times K_2$ where

$$K_1 = \{A_1 \in \mathbb{R}: \tfrac{1}{2} \leqslant A_1 \leqslant \tfrac{3}{2}\}$$

$$K_2 = \{A_2 \in \mathbb{C}: |A_2| \geqslant 1\} \cup \{\infty\},$$

and let $f: K \times \mathbb{R} \to \bar{\mathbb{C}}$ be defined by

$$f(A_1, A_2, s) = 10A_2^2 + A_1(3s - 10)A_2 + sA_1.$$

Let V be the zero set of f and take $A_2 \in \operatorname{int} K_2$ in the main theorem. The objective is to find the set of points s on ∂V corresponding to this case.

Note that

$$\xi_1 \in (\tfrac{1}{2}, \tfrac{3}{2}) \tag{12}$$

$$\xi_2^2 + \xi_3^2 > 1 \tag{13}$$

and the real and imaginary parts of f are

$$f_1(A(\xi_1, \xi_2, \xi_3), s) = 10(\xi_2^2 - \xi_3^2) + \xi_1(3s - 10)\,\xi_2 + s\xi_1 = 0 \tag{14}$$

$$f_2(A(\xi_1, \xi_2, \xi_3), s) = \xi_3(20\xi_2 - (10 - 3s)\xi_1) = 0. \tag{15}$$

Denote $f = (f_1, f_2)$.

This problem arose when we used our method of "zero sets location" (based on the main theorem), to robustly stabilize by output feedback a discrete system under uncertainty conditions [37]. The function f represents the characteristic polynomial, s is the searched design parameter of the feedback, A_1 represents the uncertainty parameter, and A_2 represents the complex discrete frequency.

Applying the main theorem, we obtain that the system (3) is reduced here to

$$\frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_2)} = \xi_3((10 - 3s)^2\,\xi_1 - 20s) = 0 \tag{16}$$

$$\frac{\partial(f_1, f_2)}{\partial(\xi_2, \xi_3)} = ((10 - 3s)\,\xi_1 - 20\xi_2)^2 + (20\xi_3)^2 = 0 \tag{17}$$

$$\frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_3)} = (20\xi_2 - (10 - 3s)\,\xi_1) \times ((3s - 10)\,\xi_2 + s) + 20\xi_3^2(10 - 3s) = 0. \tag{18}$$

It is easy to see that equations arrived from the main theorem are dependent. Moreover, consider the Jacobian matrix of f, i.e.,

$$f' = \begin{bmatrix} \xi_2(3s - 10) + s & 20\xi_2 + (3s - 10)\xi_1 & -20\xi_3 & s \\ -\xi_3(10 - 3s) & 20\xi_3 & 20\xi_2 - (10 - 3s)\xi_1 & 3\xi_1\xi_3 \end{bmatrix}.$$

At points $(A(\xi_1, \xi_2, \xi_3), s)$ where $f = 0$ and (12)–(18) are satisfied we have

$$1 = \operatorname{rank} f'(A, s) < 2,$$

therefore, Theorem 1 is also not applicable in this case (see discussion before Theorem 2). Let us see how the problem can be solved by using Theorem 2.

Using the notations of Theorem 2, it is easy to see that

$$\xi^1 = (\xi_2, \xi_3) = \left(\frac{10 - 3s}{20}\,\xi_1, 0\right) \tag{19}$$

satisfies the equation $f_2=0$ in each neighborhood of (ξ_1, s). Also this solution is unique. Substituting (19) in $f_1=0$ will imply

$$f_1\left(A\left(\xi_1, \frac{10-3s}{20}\xi_1, 0\right), s\right)$$
$$= 10\left(\frac{10-3s}{20}\xi_1\right)^2 + (3s-10)\frac{10-3s}{20}\xi_1^2 + s\xi_1 = 0.$$

Dividing by ξ_1 (recall that $\xi_1 \neq 0$ since $\xi_1 \in (1/2, 3/2)$) we conclude that the equations which correspond to (8) and (9) in Theorem 2 are

$$g(\xi_1, s) = 10\left(\frac{10-3s}{20}\right)^2 \xi_1 + (3s-10)\frac{10-3s}{20}\xi_1 + s = 0 \tag{20}$$

and

$$\frac{\partial g}{\partial \xi_1} = 10\left(\frac{10-3s}{20}\right)^2 + (3s-10)\frac{10-3s}{20} = 0. \tag{21}$$

The equations (20)–(21) cannot be satisfied for the domain (13). Therefore, the set of points s on ∂V corresponding to this case is empty.

In these examples $d=2$. The treatment for general d is similar.

IV. Stabilization of Systems with Uncertainty

In this section we use the results of the previous sections in solving problems related to the *robustness of the stability of multivariable linear feedback systems* in the presence of plant and/or compensator perturbations. The focus is on *designing stabilizing compensators* for imprecisely known plants. Thus a nominal plant description is available, together with a description of the plant uncertainty, and the objective is to design a compensator that stabilizes *all* plants lying within the specified band of uncertainty. Suppose further that the plant uncertainty can be modeled by considering a *family* of plants, where the uncertainty parameter assumes values in $\mathbb{R}^n$, for some integer n. Typically the uncertainty parameter represents some physical parameter of the plant. Using our method of *zero sets location* we provide the *complete* design parameter space, which allows robust stabilization of a system under uncertainty conditions, by output (or state) feedback; i.e., the closed loop systems of the family of all plants lying within the specified band of uncertainty, is guaranteed to become stable. Moreover, the system may be continuous or discrete, and the stabilization may, if desired, be "relative," i.e., with safety margins. (The desired domain in the complex plane, where we wish to cluster the zeros of the characteristic polynomial, is arbitrary.) In addition, since the objective of compensator design is not always merely to stabilize a plant but also to improve its response, such a result contributes also in this context. Needless to mention that the ability to determine the complete design parameter space, provides *necessary and sufficient conditions for robust stabilizability of the multivariate linear feedback system*, by output (or state) feedback, at least in an algorithmic way. The system is robustly stabilizable by output (or state) feedback if, and only if, the set of feasible design parameters is not empty.

In the following we will show how the results of the previous sections can be applied to solve such robust stabilization problems. We will use a vehicle the following example, from which the general method will be evident. As we will see, the solution for the example becomes possible due to the new results, Theorems 1 and 2, of this paper.

A. *Robust Stabilization in the Presence of Plant Perturbations*

Consider first the nominal linear time-invariant multivariable system [9, 10],

$$\dot{x} = \begin{bmatrix} r_1^0 & 0 \\ 0 & r_2^0 \end{bmatrix} x + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} u, \qquad y = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} x,$$

where $r_1^0 = -1$ and $r_2^0 = -2$. Now, assume that the entries r_1^0 and r_2^0 are subject to perturbations such that $|\Delta r_1| \leqslant 1$ and $|\Delta r_2| \leqslant 1$, i.e., $r_i^0 - 1 \leqslant r_i \leqslant r_i^0 + 1$, $i = 1, 2$. It is desired to determine all the decoupled output feedback controllers

$$u = \begin{bmatrix} -s_1 & 0 \\ 0 & -s_2 \end{bmatrix} y$$

which stabilize the closed-loop family of perturbed systems. In other words we want to determine the complete design parameter space in the plane (s_1, s_2). This is a special case of the design problem formulated in Section I. This example is deliberately chosen to be sufficiently simple (but not trivial) to allow checking our final results by ad hoc nonsystematic calculations.

The state equation for the perturbed closed-loop system will have the form

$$\dot{x} = A_p(r_1, r_2, s_1, s_2) x,$$

where $r_1 \in [-2, 0]$, $r_2 \in [-3, -1]$, and

$$A_p = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} \begin{bmatrix} -s_1 & 0 \\ 0 & -s_2 \end{bmatrix} \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix}.$$

The objective is to determine the complete set S of all points (s_1, s_2) in $\mathbb{R}^2$ for which the roots λ of the characteristic polynomial

$$\begin{aligned} f(\lambda, r_1, r_2, s_1, s_2) &= \det(\lambda I - A_p(r_1, r_2, s_1, s_2)) \\ &= \lambda^2 + \lambda(-47s_1 + 50s_2 - r_1 - r_2) + 2s_1 s_2 \\ &\quad + r_1(96s_1 - 98s_2) + r_2(-49s_1 + 48s_2) + r_1 r_2 \end{aligned}$$

lie in the open left half complex plane $\operatorname{Re} \lambda < 0$, for all values of $r_1 \in [-2, 0]$ and $r_2 \in [-3, -1]$. Let us denote

$$\Lambda = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant 0\} \cup \{\infty\},$$

then S may be described by

$$\begin{aligned} S = \{(s_1, s_2) \in \mathbb{R}^2 : f(\lambda, r_1, r_2, s_1, s_2) \neq 0, \ \forall \lambda \in \Lambda \text{ and} \\ \forall (r_1, r_2) \in [-2, 0] \times [-3, -1]\}. \end{aligned}$$

Note that in the case of stability of discrete systems or the case of relative stability, the only difference is in the definition of Λ.

The complement of S in $\mathbb{R}^2$ denoted by V, becomes

$$\begin{aligned} V = \{(s_1, s_2) \in \mathbb{R}^2 : \exists \lambda \in \Lambda, \exists (r_1, r_2) \in [-2, 0] \times [-3, -1] \\ \text{such that } f(\lambda, r_1, r_2, s_1, s_2) = 0\} \end{aligned}$$

which is readily recognized as a zero set of the complex-valued function f. Hence the problem of robust stabilization of the perturbed multivariable linear feedback system is reduced to the problem of locating the zero set V. Since f is continuously differentiable in the real sense in $\Lambda \times K_1 \times K_2 \times \mathbb{R}^2$, where

$$r_1 \in K_1 = [-2, 0] \qquad \text{and} \qquad r_2 \in K_2 = [-3, -1],$$

and since Λ, K_1, and K_2 are closed sets in $\bar{\mathbb{C}}$ with smooth simple boundaries, the theorems presented in the previous sections and, consequently, the procedure of determining V, are applicable. As outlined in the procedure, first one finds the set L. In the present example, $L = L_0 \cup L_1 \cup L_2 \cup L_3$. To illustrate the methods we will detail in the Appendix the derivation of L_0. The sets L_1, L_2, L_3 can be derived similarly. The set L, which is a one-dimensional set, is depicted in Fig. 1.

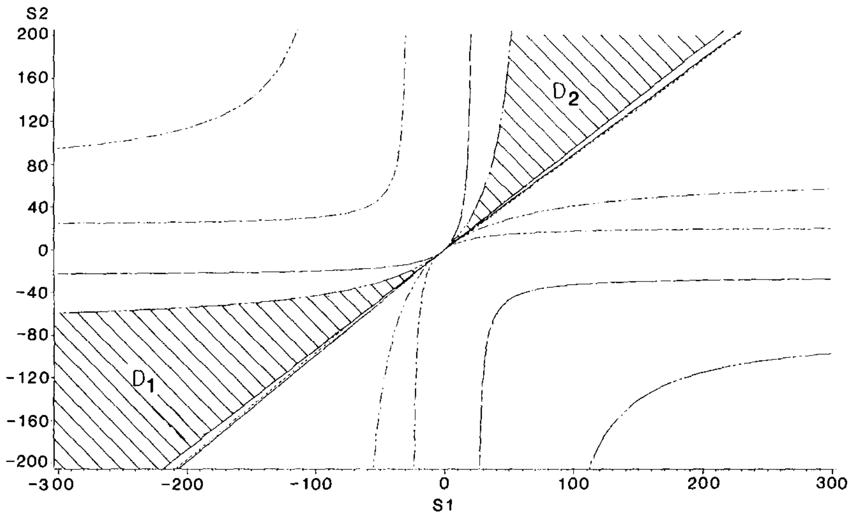

FIG. 1. The complement of the zero set of Section IV-A.

The curve L divides $\mathbb{R}^2$ into 35 connected domains D_i. As explained in Section II, in order to decide which of the domains D_i belongs to V, we choose arbitrary points in each of the domains D_i and check whether these points belong to V. Methodically, this task amounts to the original problem with reduced dimensionality, which can be carried out iteratively until the final solution is reached. In the present example, it is readily verified that the complement of V in $\mathbb{R}^2$, denoted by S, is the union $D_1 \cup D_2$ which is dashed in Fig. 1. Explicitly, the set in the (s_1, s_2) plane such that for every point (s_1^0, s_2^0) in this set, and only for these points, all the roots of the characteristic polynomial $f(\lambda, r_1, r_2, s_1^0, s_2^0)$ of the closed-loop system are confined to the open left half complex plane, for all values of (r_1, r_2) in $[-2, 0] \times [-3, -1]$, is

$$
\begin{aligned}
S = \{(s_1, s_2) \in \mathbb{R}^2 \colon & s_2 + 147s_1/(-144 + 2s_1) < 0 \text{ and} \\
& s_2 - (-2 + 143s_1)/(148 + 2s_1) > 0 \text{ and} \\
& s_2 + 0.02 - 0.94s_1 > 0 \text{ and } 49s_1 - 48s_2 + 2s_1s_2 > 0\}.
\end{aligned}
$$

B. *Design Centering*

Obviously, having the complete feasible set of controller's gains (s_1, s_2) enables us to choose the "best" nominal controller. If no other constraints and design requirements are specified for the controller, the best one would be at a "center" point which allows the maximal controller perturbations without impairing the stability of the system.

Qiu and Davison [9] considered the system treated above, assuming no uncertainty of the plant ($r_1 = r_1^0 = -1$ and $r_2 = r_2^0 = -2$) and a given nominal controller ($s_1 = s_2 = 1$). They found that the maximal controller's perturbation ε (such that $|\Delta s_1| < \varepsilon/2$ and $|\Delta s_2| < \varepsilon$) which guarantees closed loop stability is $\varepsilon = 0.0816$, and thus the system has a very small gain-margin tolerance. In this section we show that this value of ε is consistent with the results obtained by the method presented in this paper. However, a much better nominal controller can be selected, considerably enlarging the gain-margin tolerance of the system. Of course, in this simple example the results can be reached by other nonsystematic ways as well. However, the example serves to illustrate the generality of our method.

Regressing from the general results obtained in Section IV-A, let $r_1 = r_1^0 = -1$ and $r_2 = r_2^0 = -2$. Then, the state equation for the perturbed closed-loop system will have the form

$$\dot{x} = A_p^*(s_1, s_2)x,$$

where $(s_1, s_2) \in \mathbb{R}^2$ and

$$A_p^*(s_1, s_2) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} \begin{bmatrix} -s_1 & 0 \\ 0 & -s_2 \end{bmatrix} \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix}.$$

The objective is to determine the set S^* of all points (s_1, s_2) in $\mathbb{R}^2$ for which the roots λ of the characteristic polynomial

$$\begin{aligned} f^*(\lambda, s_1, s_2) &= \det(\lambda I - A_p^*(s_1, s_2)) \\ &= \lambda^2 + \lambda(-47s_1 + 50s_2 + 3) + 2s_1 s_2 \\ &\quad - (96s_1 - 98s_2) - 2(-49s_1 + 48s_2) + 2 \end{aligned}$$

lie in the left half complex plane $\operatorname{Re} \lambda < 0$. Obviously, this set must include the set S found in Section IV-A. Denote

$$\Lambda = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant 0\} \cup \{\infty\},$$

then S^* may be described by

$$S^* = \{(s_1, s_2) \in \mathbb{R}^2 : f^*(\lambda, s_1, s_2) \neq 0, \ \forall \lambda \in \Lambda\}.$$

Determination of S^* will provide us with the *complete* space (not necessarily rectangular) of permissible perturbations in the gains of the controller, which maintain stability.

The complement of S^* in $\mathbb{R}^2$ denoted by V^*, becomes

$$V^* = \{(s_1, s_2) \in \mathbb{R}^2 : \exists \lambda \in \Lambda \text{ such that } f^*(\lambda, s_1, s_2) = 0\}$$

which is readily recognized as a zero set of the complex-valued function f^*. Hence the problem of finding the complete space of permissible perturbations in the gains of the controller, of the multivariable linear feedback system, is again reduced to the problem of locating the zero set V^*. Obviously, $V^* \subset V$, and the procedure for finding V^* is shorter. Indeed, the set L, in this case is the union, $L = L_0 \cup L_1$. This set is depicted in Fig. 2. The curve L divides $\mathbb{R}^2$ into eight connected domains D_i. It is readily verified that the complement of V^* in $\mathbb{R}^2$, denoted by S^*, is the union $D_1^* \cup D_2^*$ which is dashed in Fig. 2. Explicitly, the set in the (s_1, s_2) plane such that for every point (s_1^0, s_2^0) in this set, and only for these points, all the roots of the characteristic polynomial $f^*(\lambda, s_1^0, s_2^0)$ of the closed-loop system are confined to the open left half complex plane, is

$$S^* = \{(s_1, s_2) \in \mathbb{R}^2 \colon s_1 > -1 \text{ or } s_2 < -1 \text{ and } s_2 + 0.06 - 0.94 s_1 > 0\}.$$

In comparison, the results in [9] for this same example provide the rectangle (depicted not to scale in Fig. 2)

$$\mathscr{R}_{QD} = \{(s_1, s_2) \in \mathbb{R}^2 \colon |s_1 - 1| < \varepsilon/2,\ |s_2 - 1| < \varepsilon\} \subset S^*,$$

where $\varepsilon = 0.0816$. This result is consistent with our result.

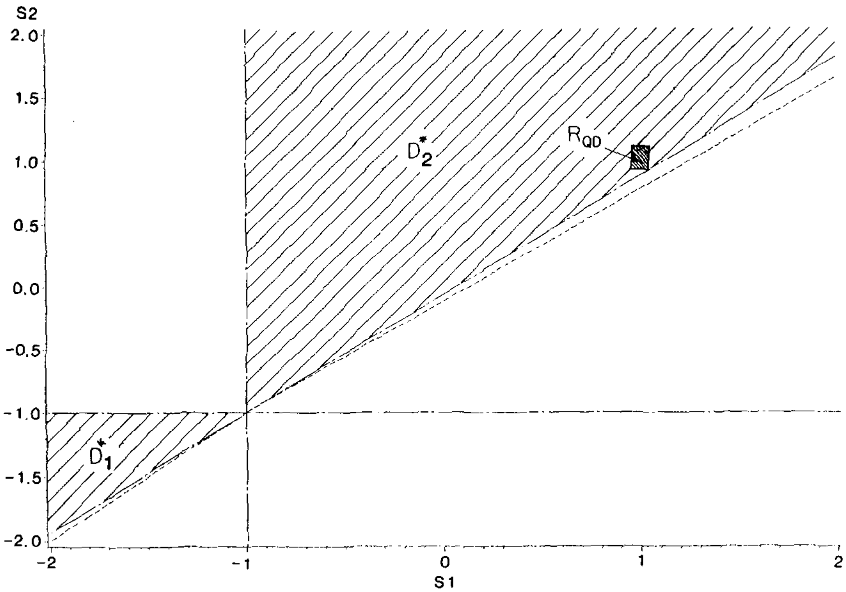

FIG. 2. The complement of the zero set of Section IV-B.

V. Conclusion

The method established in [1, 3] of locating zero sets is generalized here to continuously differentiable scalar and vector-valued functions, which depend on several real variables and complex parameters. The first stage in locating a zero set V, according to the method, is to locate an $(m-1)$-dimensional set L, which contains the boundary of V. The set L is derived from equations which represent necessary conditions on the boundary of the zero set. Such conditions are provided here and in [3] by the main theorem. The main theorem, which in some cases does not provide sufficiently many independent equations, is supported now by two new theorems: Theorems 1 and 2. These theorems provide a method of generating new additional equations for the boundary of V, which together with the equations of the main theorem are capable of yielding a suitable $(m-1)$-dimensional set L, in cases where the main theorem alone was unable to do so or yielding inconvenient equations.

The extensions and improvements in the method of locating zero sets enable us now to get a complete solution to additional robust design problems as illustrated in this paper.

Appendix

Derivation of the Set L_0 for the Problem of Section IV-A

The set L_0 in $\mathbb{R}^2$ is the union of the one-dimensional sets in $\mathbb{R}^2$ which corresponds to the cases $l=2$ and $l=3$. The cases $l=0$ and $l=1$ are not applicable, since int $K_1=\varnothing$ and int $K_2=\varnothing$.

Let us first consider the case $l=2$. Then

$$\lambda=(x+jy)\in \operatorname{int} \Lambda, \qquad -2<r_1<0 \text{ and } -3<r_2<-1 \tag{A1}$$

and (2)–(3) of the main theorem reduce to

$$\begin{aligned}
\operatorname{Re} f &= f_1(x, y, r_1, r_2, s_1, s_2)\\
&= x^2-y^2+x(-47s_1+50s_2-r_1-r_2)\\
&\quad + r_1(96s_1-98s_2)+r_2(-49s_1+48s_2)+2s_1s_2+r_1r_2=0\\
\operatorname{Im} f &= f_2(x, y, r_1, r_2, s_1, s_2)=y(2x-47s_1+50s_2-r_1-r_2)=0\\
f_3 &= \frac{\partial(f_1, f_2)}{\partial(x, y)}\\
&= \begin{vmatrix} 2x-47s_1+50s_2-r_1-r_2 & -2y\\ 2y & 2x-47s_1+50s_2-r_1-r_2\end{vmatrix}=0.
\end{aligned}$$

The equations $\partial(f_1, f_2)/\partial(x, r_1)=0$, $\partial(f_1, f_2)/\partial(x, r_2)=0$, $\partial(f_1, f_2)/\partial(y, r_1)=0$, $\partial(f_1, f_2)/\partial(y, r_2)=0$, and $\partial(f_1, f_2)/\partial(r_1, r_2)=0$ are included in $f_3=0$. Evidently, this is a case where the main theorem yields *dependent* equations; in order to solve the problem we need additional equations. Hence we apply Theorem 2. It is easy to see that

$$(x, y)=\left(\frac{47s_1-50s_2+r_1+r_2}{2}, 0\right) \tag{A2}$$

satisfies the vector equation $v=(f_2, f_3)=0$ in each neighborhood of (r_1, r_2, s_1, s_2). Also this solution is unique. Substituting this solution in $f_1=0$ and setting

$$r=(r_1-r_2),$$

we obtain that the equations which correspond to (10) and (11) in Theorem 2 are reduced to

$$\begin{aligned} g(r, s_1, s_2) &= 1177s_1s_2+72.5s_1r-73s_2r-0.25r^2 \\ &\quad -552.25s_1^2-625s_2^2=0 \\ \frac{\partial g}{\partial r} &= 72.5s_1-73s_2-0.5r=0. \end{aligned}$$

The last two equations together with (A1) and remembering that x of (A2) should be positive, yield the set

$$\{(s_1, s_2)\in\mathbb{R}^2: s_1=s_2, s_1\in(-3, -5/3)\}. \tag{A3}$$

Now consider the case $l=3$. Here

$$\lambda=jy\in\partial\Lambda\setminus\{\infty\}, \qquad -2<r_1<0 \text{ and } -3<r_2<-1. \tag{A4}$$

Using only the part of the main theorem equations which correspond to (2) we obtain

$$\begin{aligned} \operatorname{Re} f &= f_1(y, r_1, r_2, s_1, s_2) \\ &= -y^2+r_1(96s_1-98s_2)+r_2(-49s_1+48s_2)+2s_1s_2+r_1r_2=0 \\ \operatorname{Im} f &= f_2(y, r_1, r_2, s_1, s_2)=y(-47s_1+50s_2-r_1-r_2)=0. \end{aligned}$$

Now consider the solution

$$r_1=-47s_1+50s_2-r_2$$

of $f_2=0$. Substituting this solution in $f_1=0$, and applying Theorem 2 we obtain

$$g(y, r_2, s_1, s_2) = -y^2 + 9408 s_1 s_2 - 192 s_1 r_2 + 196 s_2 r_2 - 4512 s_1^2 - 4900 s_2^2 - r_2^2 = 0$$

$$\frac{\partial g}{\partial y} = -2y = 0$$

$$\frac{\partial g}{\partial r_2} = -192 s_1 + 196 s_2 - 2 r_2 = 0.$$

The last three equations together with (A4) yield the set

$$\{(s_1, s_2) \in \mathbb{R}^2 : s_1 = s_2, s_1 \in (-3/2, -1/2)\}. \tag{A5}$$

The other solution of $f_2=0$ ($y=0$) yields the same set (A5).

References

1. E. Zeheb and E. Walach, Zero sets of multiparameter functions and stability of multidimensional systems, *IEEE Trans. Acoust. Speech Signal Process.* **ASSP-29** (1981), 197–206.
2. E. Walach and E. Zeheb, Generalized zero sets of multiparameter polynomials and feedback stabilization, *IEEE Trans. Circuits and Systems* **CAS-29** (1982), 15–23.
3. G. Fruchter, U. Srebro, and E. Zeheb, On several variable zero sets and application to MIMO robust feedback stabilization, *IEEE Trans. Circuits and Systems* **CAS-34** (1987), 1208–1220.
4. E. Zeheb and D. Hertz, Robust control of the characteristic values of systems with possible parameter variations, *Internat. J. Control* **40** (1984), 81–96.
5. J. N. Chiasson, S. D. Brierley, and E. Lee, A simplified derivation of the Zeheb–Walach 2-D stability test with applications to time-delay systems, *IEEE Trans. Automat. Control* **AC-30** (1985), 411–414.
6. D. Hertz, E. I. Jury, and E. Zeheb, Stability independent and dependent of delay for delay differential systems, *J. Franklin Inst.* **318**, No. 3 (1984), 143–150.
7. E. Zeheb and P. Landau, A new approach to design of coupling networks, *J. Franklin Inst.* **318**, No. 5 (1984), 357–371.
8. P. Landau and E. Zeheb, On solving broadband matching design equations, *Internat. J. Circuit Theory Appl.* **13** (1985), 123–132.
9. L. Qiu and E. J. Davison, New perturbation bounds for robust stability of linear state space models, *in* "Proceedings, 1986 IEEE Conf. Decision Contr., Athens, Greece, December 1986," 751–755.
10. E. J. Davison and B. R. Copeland, Gain margin and time lag tolerance constraints applied to the stabilization problem and robust servomechanism problem, *IEEE Trans. Automat. Control* **AC-30** (1985), 229–239.
11. M. J. Chen and C. A. Desoer, Necessary and sufficient conditions for robust stability of linear distributed feedback systems, *Internat. J. Control* **35** (1982), 255–267.

12. J. C. DOYLE AND G. STEIN, Multivariable feedback design: Concepts for a classical/modern synthesis, *IEEE Trans. Automat. Control* **AC-26** (1981), 4–16.
13. J. DOYLE, Analysis of feedback systems with structured uncertainties, *IEE Proc. Part D* **129** (1982), 242–250.
14. M. G. SAFONOV, Stability margins of diagonally perturbed multivariable feedback systems, *IEE Proc Part D* **129** (1982), 251–256.
15. M. VIDYASAGAR AND H. KIMURA, Robust controllers for uncertain linear multivariable systems, *Automatica* **22**, No. 1 (1986), 85–94.
16. J. M. MARTIN, State-space measures for stability robustness, *IEEE Trans. Automat. Control* **AC-32** (1987), 509–512.
17. C. B. SOH, C. S. BERGER, AND K. P. DABKE, On the stability properties of polynomials with perturbed coefficients, *IEEE Trans. Automat. Control* **AC-30** (1985), 1033–1036.
18. C. B. SOH, C. S. BERGER, AND K. P. DABKE, Addendum to "on the stability properties of polynomials with perturbed coefficients," *IEEE Trans. Automat. Control* **AC-32** (1987), 239–240.
19. R. V. PATEL AND M. TODA, Quantitative measures of robustness for a multivariable system, *in* "Proceedings of Joint Automatic Control Conference, San Francisco, 1980."
20. R. K. YEDAVALLI, Perturbation bounds for robust stability in linear state space models, *Internat. J. Control* **42**, No. 6 (1985), 1507–1517.
21. K. M. ZHOU AND P. P. KHARGONEKAR, Stability robustness bounds for linear state space models with structured uncertainty, *IEEE Trans. Automat. Control* **AC-32**, No. 7 (1987), 621–623.
22. M. FU AND B. R. BARMISH, Maximal unidirectional perturbation bounds for stability of polynomials and matrices, to be published.
23. G. LEITMANN, E. P. RYAN, AND A. STEINBERG, Feedback control of uncertain systems: Robustness with respect to neglected actuator and sensor dynamics, *Internat. J. Control* **43**, No. 4 (1986), 1243–1256.
24. B. R. BARMISH, Invariance of the strict Hurwitz property for polynomials with perturbed coefficients, *IEEE Trans. Automat. Control* **AC-29**, No. 10 (1984), 935–936.
25. J. P. GUIVER AND N. K. BOSE, Strictly Hurwitz property invariance of quartics under coefficient perturbation, *IEEE Trans. Automat. Control* **AC-28** (1983), 106–107.
26. S. BIALAS AND J. GARLOFF, Stability of polynomials under coefficient perturbation, *IEEE Trans. Automat. Control* **AC-30** (1985), 310–312.
27. N. K. BOSE, E. I. JURY, AND E. ZEHEB, On robust Hurwitz and Schur polynomials, *IEEE Trans. Automat. Control* **AC-33** (1988), 1166–1168.
28. E. I. JURY AND E. ZEHEB, An algebraic algorithm for determining the desired gain of multivariable feedback systems, *Internat. J. Control* **37** (1983), 1081–1094.
29. J. E. ACKERMAN, Parameter space design of robust control systems, *IEEE Trans. Automat. Control* **AC-25** (1980), 1058–1072.
30. G. LEITMANN, On efficacy of nonlinear control in uncertain linear systems, *J. Dynam. Syst. Measurement Control* **102** (1981), 95–102.
31. B. R. BARMISH, M. CORLESS, AND G. LEITMANN, A new class of stabilizing controllers for uncertain dynamical systems, *SIAM J. Control Optim.* **21** (1983), 246–255.
32. H. KWAKERNAAK, Uncertainty models and the design of robust control systems, *in* "Uncertainty and Control" (J. Ackermann, Ed.), pp. 84–107, Lecture Notes in Control and Information Sciences, Vol. 70, Springer-Verlag, New York/Berlin, 1985.
33. M. CORLESS, Stabilization of uncertain linear systems, *in* "Proceedings of the IFAC Workshop on Model Error Concepts and Compensation, Boston, 1985."
34. I. R. PETERSON AND C. V. HOLLOT, A Riccati equation approach to the stabilization of uncertain linear systems, *Automatica* **22**, No. 4 (1986), 369–411.

35. C. V. HOLLOT AND M. ARABACIOGLU, 1 step Lyapunov min-max controllers, *in* "Proceedings of the ACC, Minneapolis, MN, 1987," pp. 496–501.
36. G. LEITMANN AND E. P. RYAN, Output feedback control of a class of singularly perturbed uncertain dynamical systems, *in* "Proceedings of the ACC, Minneapolis, MN, 1987," pp. 1590–1594.
37. G. FRUCHTER, "Zero Sets Location and Robust Stabilization of Control Systems," Ph.D. dissertation, Technion-Israel Institute of Technology, Haifa, Israel, March 1988.
38. G. FRUCHTER, U. SREBRO, AND E. ZEHEB, An analytic method for design of uncertain discrete nonlinear control systems, *in* "Proceedings of the 15th Conf. of IEEE in Israel, 1987."